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Nonlinear Rayleigh-Taylor growth in converging geometry

Daniel S. Clark* and Max Tabak
Lawrence Livermore National Laboratory, University of California, Livermore, California 94550, USA
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The early nonlinear phase of Rayleigh-Taylor growth is typically described in terms of the classic Layzer model in which bubbles of light fluid rise into the heavy fluid at a constant rate determined by the bubble radius and the gravitational acceleration. However, this model is strictly valid only for planar interfaces and hence ignores any effects that might be introduced by the spherically converging interfaces of interest in inertial confinement fusion and various astrophysical phenomena. Here, a generalization of the Layzer nonlinear bubble rise rate is given for a self-similar spherically converging flow of the type studied by Kidder. A simple formula for the bubble amplitude is found showing that, while the bubble initially rises with a constant velocity similar to the Layzer result, during the late phase of the implosion, an acceleration of the bubble rise rate occurs. The bubble rise rate is verified by comparison with numerical hydrodynamics simulations.

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Applications in inertial confinement fusion (ICF) and a variety of astrophysical phenomena have motivated intensive investigations of the Rayleigh-Taylor (RT) instability, the unstable acceleration of a heavy fluid by a light fluid ([1] and references therein). The early nonlinear stage of the RT instability has traditionally been described in terms of the Layzer model [2] in which bubbles of light fluid rise into the heavy fluid with a constant velocity while spikes of heavy fluid fall into the light fluid with constant acceleration. While Layzer's model simply and quite accurately describes the nonlinear phase of RT growth prior to turbulent mixing, it is strictly valid only for planar fluid interfaces. Given that many applications involve either spherically converging or diverging flows, it is relevant to consider how the nonlinear growth phase might be modified at a spherical interface. Extensive numerical modeling of instability growth on spherical interfaces has been performed, e.g., [3], but without a theoretical foundation similar to that provided in the planar case by the Layzer model. Here an analytical calculation of the bubble rise rate is presented for the spherically convergent RT instability. Convergence is shown to enhance nonlinear RT growth compared to the planar case. Beyond revealing this important physical effect and the scaling properties introduced by sphericity, uncovering the spherical analogue of the Layzer model also provides a rigorous and relevant potential test problem for hydrodynamics simulations of spherical flows.

The Layzer model treats the confining effect of neighboring RT bubbles as an effective cylindrical boundary condition enclosing a rotationally symmetric central bubble. The corresponding spikes of heavy fluid run down the cylindrical walls. Considering an incompressible and irrotational flow, the fluid velocity may be described as the gradient of a potential function ϕ given by a solution of Laplace's equation. For a light fluid of infinitesimal density (Atwood number of $\mathcal{A}=1$), the boundary conditions on the bubble surface are that the pressure be uniform and that the interface (denoted by $S=0$ below) move with the flow. The remaining boundary conditions are that the fluid be at rest at infinity and that there be no flow through the cylindrical walls. A considerable simplification is possible for the planar problem: Since the bubble rises *a posteriori* with a constant velocity, it is convenient to transform the problem to a frame moving with the bubble so that the flow near the bubble apex is static. By keeping only the lowest order mode in the expansion of the velocity potential in this frame and expanding the Bernoulli integral on the bubble surface to second order in the radial distance from the cylindrical axis, a solubility condition of this equation gives the bubble velocity as $u \simeq \sqrt{g\,r_0/k_0}$. Here, g is the inertial acceleration of the interface or an effective gravitational acceleration, r_0 is the bubble radius or perturbation wavelength, and k_0 is the first root of the Bessel function J_1. Since this solution includes only the lowest mode of the velocity potential and is carried only to second order in the distance from the axis, only the flow near the axis is described and not the behavior of the spikes near the walls. Nevertheless, the theoretical value for u compares quite favorably with numerical simulations and experimental measurements. The effects of compressibility, bubble merging, density gradients, $\mathcal{A}<1$, Kelvin-Helmholtz roll up, etc., are not included in the model.

To adapt Layzer's model to spherical interfaces, the first modification is to replace the cylindrical coordinates and boundary conditions by spherical coordinates and conical boundary conditions to capture the lowest order effect of spherical convergence. The problem is immediately complicated by this modification in that the bubble no longer rises with a constant velocity and transforming to a frame where the flow near the bubble apex is static is no longer trivial. The calculation must then be carried through for a time-dependent flow and a time-evolving interface. Indeed, calculating the rate of rise of a bubble enclosed by a narrow cone opening downward in a uniform gravitational field shows that the bubble nonlinearly decelerates as it rises. This result comports with the Layzer model in that, as the bubble rises, the effective radius enclosing the bubble shrinks and the bubble must then rise at an ever slower velocity according to the Layzer formula.

*Electronic address: clark90@llnl.gov

More importantly, however, a uniform gravitation field is no longer precisely equivalent to a uniform acceleration on a spherical interface. A uniform gravitational field in the spherical case is the equivalent of setting the entire sphere into accelerated motion which is not relevant to the instabilities of the imploding interface. In place of a uniform gravity, one consistent representation of an accelerated spherical interface is to consider the growth of perturbations in a spherical coordinate system accelerating smoothly toward the origin, i.e., transform to a (primed) frame moving with respect to the fixed (unprimed) frame according to $\{r'=r/h(t),\ \theta'=\theta,\ t'=t\}$. Here $h(t)$ is the scale factor describing the radial contraction of the primed coordinates with respect to the unprimed coordinates. The character of this transformation is readily identified with that of a self-similar spherically converging flow, and the notation is motivated in connection with the flow studied by Kidder [4]. Under this transformation, the velocity potential, Bernoulli's equation, and the equation of motion for the bubble surface S become

$$\phi' = h^{-2}\phi + \frac{(r')^2}{2}\frac{\dot h}{h} + \Phi(t'),$$

$$F(t') = \left. \phi'_{t'} - \frac{|\vec\nabla'\phi'|^2}{2} - \frac{\gamma}{\gamma-1}\frac{p'}{\rho'} + 2\frac{\dot h}{h}\phi' - \frac{(r')^2}{2}\frac{\ddot h}{h}\right|_{S'=0},$$

$$0 = S'_{t'} - \vec\nabla'\phi' \cdot \vec\nabla' S'. \tag{1}$$

Here subscripts denote partial differentiation, dots denote total derivatives with respect to time, and $\Phi(t')$ and $F(t')$ are arbitrary functions of time. The inertial terms appearing in the Bernoulli integral play the role of gravitational potentials in the interface frame but correctly incorporate the spherical nature of the flow. Consistent with the spherical convergence of the flow, compressibility of the fluid is allowed in this model with γ the usual ratio of specific heats. This is a distinction from the incompressible Layzer model. Forbidding compressibility in converging flows (as in the linear model of Plesset [5]), necessitates unphysical singularities of the velocity field at $r=0$.

To develop an analog of Layzer's analysis, we solve these equations in a second-order neighborhood of the bubble apex (i.e., to $\mathcal{O}(\theta^2)$ in spherical polar coordinates aligned with the bubble) and include only the lowest order mode in the expansion of the velocity potential in the interface frame. The velocity potential ansatz in the primed frame in spherical coordinates is

$$\phi' = A(t')(r')^{\nu}P_{\nu}(\cos\theta')$$

$$\Leftrightarrow \phi = -\frac{r^2}{2}\frac{\dot h}{h} + \tilde A(t) r^{\nu} P_{\nu}(\cos\theta). \tag{2}$$

Here $A(t)$ is the time-dependent nonlinear perturbation amplitude to be determined and ν is the spherical mode number determined by the boundary condition that there be no flow through the cone walls. The function $\Phi(t')$ can be incorporated into $F(t')$ without loss of generality. Note that in the fixed (unprimed) frame, the velocity potential is separated into a compressible component determined by the background spherically converging flow and an incompressible nonlinear perturbation. In the primed frame, however, the flow is by construction incompressible. Such a separation was initially discussed by Book and Bernstein [6] in an analysis of the linear growth of perturbations on a self-similar implosion.

Characterizing the bubble in the moving frame (henceforth dropping the primes) by $S=R(\theta,t)-r$, with $R(\theta,t)=a(t)+b(t)\theta^2+\mathcal{O}(\theta^4)$ and substituting the chosen ϕ into the last of Eqs. (1) yields

$$0 = S_t - \vec\nabla\phi\cdot\vec\nabla S = \dot a + \nu A a^{\nu-1} + \theta^2\left\{\dot b + \nu(\nu-1)Aa^{\nu-2}b + \nu(\nu+1)Aa^{\nu-2}b - \frac{\nu^2(\nu+1)}{4}Aa^{\nu-1}\right\} + \mathcal{O}(\theta^3).$$

Requiring a solution at the first two orders in θ determines the perturbation amplitude $A(t)$ and the bubble curvature $b(t)$ in terms of the bubble amplitude $a(t)$

$$A = -\frac{1}{\nu}a^{1-\nu}\dot a \quad \text{and} \quad b = \frac{\nu}{4}\frac{\nu+1}{2\nu-1}a.$$

Consistent with the self-similarity of the background flow, the bubble shape at second order is found to have a separable dependence in angle and time,

$$R(\theta,t) = a(t)\left\{1 + \frac{\nu}{4}\frac{\nu+1}{2\nu-1}\theta^2\right\} + \mathcal{O}(\theta^4). \tag{3}$$

Since the RT perturbation of the velocity potential is incompressible, the fluid density evolves only due to the radially compressing component of the flow

$$\rho = \exp\int dt \nabla^2\phi = \rho_0(\vec r_0)h^{-3}(t),$$

where $\rho_0(\vec r_0)$ is the Lagrangian value of the fluid density, i.e., density of the fluid particle at its initial location, and the integral is computed along the Lagrangian trajectory of the fluid particle. For an isentropic implosion, also $p=p_0(\vec r_0)[\rho/\rho_0(\vec r_0)]^{\gamma}$. Substituting the above results for $A(t), b(t), \rho$, and p into the Bernoulli integral on the interface $S=0$ (again in the moving frame) and expanding to $\mathcal{O}(\theta^2)$ leads at the lowest orders in θ to two coupled equations for the implosion scale factor $h(t)$ and bubble amplitude $a(t)$

$$-\frac{\nu}{2}\left(\frac{R_0}{t_c}\right)^2 h^{1-3\gamma} = a\ddot a + \left(1-\frac{\nu}{2}\right)\dot a^2 + 2\frac{\dot h}{h}a\dot a + \frac{\nu}{2}\frac{\ddot h}{h}a^2 \quad \text{for} \quad \mathcal{O}(\theta^0),$$

$$0 = a\ddot a + \frac{1-2\nu}{2}\dot a^2 + 2\frac{\dot h}{h}a\dot a - \frac{\nu}{\nu-1}\frac{\ddot h}{h}a^2 \quad \text{for} \quad \mathcal{O}(\theta^2). \tag{4}$$

The constants R_0 and t_c set the length and time scales of the implosion.

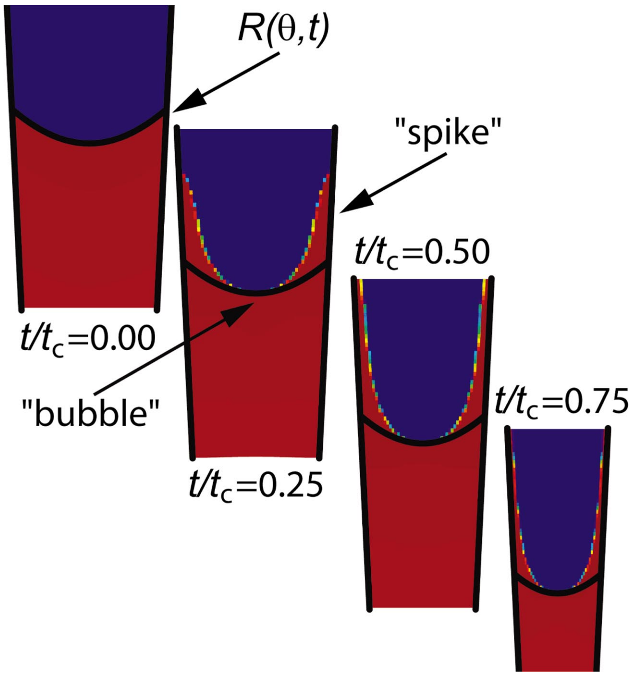

FIG. 1. (Color) Snapshots of bubble growth from a HYDRA simulation for $\nu = 80$. Red denotes the dense fluid and blue the low-density pusher. To illustrate the bubble evolution, each snapshot is centered vertically about the location of the bubble apex at the corresponding time. Near the axis of the cone, the computed bubble curvature agrees well with the theoretical prediction [Eq. (3)] shown as the thick dark line.

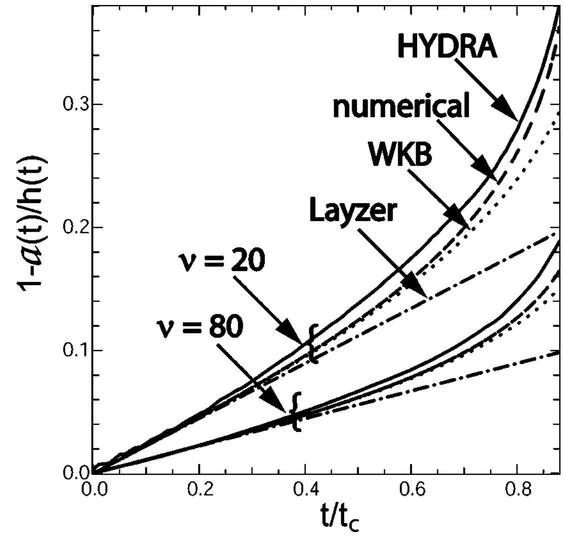

FIG. 2. Comparison of normalized bubble heights in the frame moving with the interface from the HYDRA simulation of Fig. 1, the WKB solution [Eq. (5)], the result of numerically integrating Eqs. (4), and the Layzer prediction. Results with $\nu = 20$ are also shown.

In principle, the $\mathcal{O}(\theta^2)$ equation should be solved for $a(t)$ as a functional of $h(t)$, and the result substituted into the $\mathcal{O}(\theta^0)$ equation to find a single self-consistent solution for $h(t)$. Such an exact solution to Eqs. (4) could not be found. However, the $\mathcal{O}(\theta^2)$ equation from Eqs. (4) may in general be put in Schrödinger form and, for a given slowly evolving $h(t)$, the approximate bubble amplitude calculated by the WKB method

$$a(t) \sim h^{1/(\nu-3/2)} \exp\left\{ 2\Lambda(\nu) \int^t dt \sqrt{-\ddot{h}/h} \right\},$$

with $\Lambda(\nu) \doteq \sqrt{\nu(3-2\nu)/2(1-\nu)-1}/(3-2\nu)$. This expression for $a(t)$ could be substituted into the $\mathcal{O}(\theta^0)$ equation, and an iterative approximation for $h(t)$ developed. A more tractable approach is to note that, in the limit of large ν, the $\mathcal{O}(\theta^0)$ equation reduces to Kidder's equation for the scale factor of an unperturbed self-similar implosion [4]. Specializing to the case of $\nu \gg 1$, it is then acceptable to approximate $h \simeq h_{\mathrm{Kidder}}$. For $\gamma = 5/3$, the Kidder scale factor is $h(t) = \sqrt{1-(t/t_c)^2}$ with t_c the time of total collapse to the origin of the unperturbed flow. Hence

$$a(t) \sim R_0 h^{1/(\nu-3/2)} \left(\frac{1+t/t_c}{1-t/t_c} \right)^{-1/2\sqrt{\nu}}, \quad \nu \to \infty. \tag{5}$$

Here the location of the bubble apex has been initialized to the outer radius of the sphere R_0 for the unperturbed problem. Note that, as in the unperturbed one-dimensional implosion studied by Kidder, the scale factor of the implosion $h(t)$ cannot be arbitrarily specified to generate any desired acceleration history but (due to the self-similar symmetry assumed for the flow) must be self-consistently determined by the boundary conditions and the perturbation history $a(t)$ via Eqs. (4). Though this allows only a specific acceleration history $h(t)$ for the spherical interface (approaching that of Kidder in the limit of $\nu \gg 1$), this constraint appears hardly more restrictive for potential applications than the assumption of a constant, uniform gravity is in applying the Layzer model for planar interfaces.

As should be expected, in the limit $\nu \to \infty$ (i.e., for narrow cones) and for early times $t/t_c \ll 1$ (before significant convergence has occurred), the bubble height in the moving frame and the bubble curvature reduce to those given by the Layzer model

$$1 - \frac{a(t)}{R_0} \sim \frac{1}{\sqrt{\nu}} \frac{t}{t_c} \quad \text{and} \quad \frac{1}{2} \frac{R_{\theta\theta}}{R_0} \sim \frac{\nu}{8}.$$

These results can be connected to the Layzer formulae by using the asymptotic form $P_\nu(\cos\theta) \sim J_0\big((2\nu+1)\sin(\theta/2)\big)$, $\nu \to \infty$ from which follows $\nu \sim k_0/\beta$ for $\beta \simeq r_0/R_0$ the half-angle of the cone and r_0 its effective radius. In these units, the initial acceleration of the interface is $\ddot{h}(0) = R_0/t_c^2$.

Interestingly, the formula found for the bubble amplitude is functionally similar to the result earlier found by Kidder [7] for the linear growth of perturbations during a homogeneous implosion. However, these results (the former for the distance of the bubble apex from the center of the implosion and the latter for the linear regime amplitude of a single mode perturbation on the outside of the imploding sphere) differ crucially in the exponent's dependence on the mode number: during the linear regime, the exponent scales as the square root of the perturbation mode number while as the reciprocal of the square root of the mode number during the nonlinear regime. Both cases are analogous to the mode number scalings for a planar interface. The method of calculating the bubble amplitude via the WKB technique is also superficially similar to that followed by Hattori *et al.* [8], but again this latter calculation applies only to the linear regime, and the scaling with mode number is reciprocated.

Equation (5) was verified by comparing with two-dimensional arbitrary Lagrangian-Eulerian (ALE) hydrodynamics simulations run with the HYDRA code [3]. For a given mode number ν, a simulation was initialized with slip boundary conditions on the cone walls and a Kidder-type pressure source applied through a low density pusher material (approximating $\mathcal{A}=1$) to the fluid interface. The radial density profile within the dense fluid was initialized as prescribed by Kidder [4], and the interface was nonlinearly perturbed in accordance with the initial second order bubble shape, Eq. (3). Considerable ALE relaxation of the mesh was required throughout the simulation. Care also had to be taken in initializing the proper fluid velocities according to the velocity potential Eq. (2) (in the fluid as well as the low density pusher material).

An example sequence of snapshots of bubble growth from a simulation with $\nu=80$ is shown in Fig. 1. The dense fluid is shown in red and the low-density pusher appears in blue. Mixing of the fluids due to the ALE relaxation of the mesh results in the yellow-colored boundary zones. The second-order bubble shape (as imposed at $t=0$) is denoted by the dark line. Throughout the simulation, the interface curvature at the bubble apex appears to be in good agreement with the theoretical prediction. Similar results were found for the range of mode numbers $\nu=20-160$. Since the theory is valid only to $\mathcal{O}(\theta^2)$ the growth of the spike along the wall of the cone is not captured. A perfectly analogous discrepancy applies for the description of spikes in the Layzer model.

Figure 2 illustrates the normalized bubble height as measured from Fig. 1 in comparison with the WKB solution [Eq. (5)], the result of numerically integrating Eqs. (4), and the Layzer prediction. The result of a simulation with $\nu=20$ is also shown. Following initially linear growth with time, the HYDRA, numerical, and WKB results all demonstrate substantially faster bubble growth than predicted by the Layzer model. Good agreement between the simulation results and the theoretical expectations is seen through most of the implosion. With increasing ν, closer agreement between the WKB and numerical solutions is seen for all t/t_c. Also as predicted by Eq. (5), greater acceleration of the bubble velocity over the Layzer prediction was observed for lower values of ν.

In summary, a nonlinear RT bubble model has been presented for a spherically converging flow. Consistently describing an accelerating, spherically converging interface required assuming a Kidder-type self-similar background flow within a conical boundary in place of the uniform gravity and cylindrical boundaries assumed in the Layzer model. An approximate solution was found for the growth of the bubble height indicating an initial phase with linear growth in time at the rate predicted by Layzer followed by a strong acceleration of the bubble growth rate late in the implosion. Good agreement for the bubble growth rate and curvature at the apex was found by comparison with two-dimensional hydrodynamics simulations.

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